A Short List of Important Relations for Diffusive Processes

compiled by Steven Block 10/01

Name	**Equation**	**Information**	**Comment**
Thermal energy Boltzmann's factor, *k* × absolute temperature, *T*	kT where $k =$ 1.38×10^{-16} erg/°K 1.38×10^{-23} J/°K $= R/N_{Av}$	at room temperature (298 °K): ~ 4×10^{-14} ergs ~ 4×10^{-21} J = 4 zJ ~ 4 pN·nm ~ 1/40th eV ⇒ ~0.6 kcal/mole ⇒ ~ 2.6 kJ/mole	Sets the critical energy scale for all stochastic processes in Nature!
Boltzmann relation	$\frac{N(E_1)}{N(E_2)} = e^{-\Delta E/kT}$	Specifies the relative population (i.e., relative probability) of two energy states, E_1 and E_2, at thermal equilibrium	Arguably the single most important relationship in all of statistical mechanics!
Diffusive "speeds"	$\langle x^2 \rangle = 2D_x t$ $\langle r^2 \rangle = 6D_x t$ $\langle \theta^2 \rangle = 4D_\theta t$	for motion in one dimension for motion in three dimensions for angular diffusion	Not a velocity! The r.m.s. displacement goes as the square root of time.
Einstein-Smoluchowski relation (a.k.a. Stokes-Einstein relation)	$D_r = kT/f_r$ $D_\theta = kT/f_\theta$	translational diffusion coeff. rotational diffusion coeff. *f* is the drag coefficient, defined by force $F = fv$, where *v* is velocity.	Provides vital link between the macroscopic world (*D*) and microscopic world (*f*).
Stokes drag (for a sphere)	$f_r = 6\pi\eta a$ $f_\theta = 8\pi\eta a^3$	translational drag coeff. rotational drag coeff. *a* is the radius of sphere; η is viscosity	Only holds for low Reynold's number. f_{trans} depends only on biggest dimension
Reynolds' number	$\Re = \frac{\rho a v}{\eta}$	ρ is a density, *a* is a length, *v* is a speed, η is the viscosity	Supplies the ratio of inertial forces to viscous forces

Fick's 1st law	$J = -D\frac{\partial C}{\partial x}$ $\boldsymbol{J} = -D\nabla C$	J is the flux; C is the concentration, D is the diffusion coefficient. Shown for 1D and 3D.	Flux depends on gradient; holds for concentration; temperature, too!
Equation of continuity	$\frac{\partial C}{\partial t} + \frac{\partial J}{\partial x} = 0$ $\frac{\partial C}{\partial t} + \nabla \cdot \boldsymbol{J}$	J is the flux; C is the concentration. Relates temporal and spatial changes. Shown for 1D and 3D.	Says that material is neither created or destroyed
Diffusion equation (a.k.a. Fick's 2nd Law)	$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$ $\frac{\partial C}{\partial t} = D\nabla^2 C$	This follows directly from (1) Fick's 1st Law & (2) Equation of Continuity	Generally true, but hard to solve analytically for many geometries
Diffusion with drift	$J = -D\frac{\partial C}{\partial x} + v_x C$ $\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} - v_x\frac{\partial C}{\partial x}$	The flux J at a given point x, due to velocity v_x, increases by an amount $v_x C(x)$.	Diffusion plus convection. Also called the "telegrapher's equation."
Current to capture (for a sphere)	$I = 4\pi DaC_0$ $k = 4\pi Da$	I is the current to a sphere of radius a. Therefore k is the reaction rate constant for capture (i.e., the flux per unit of concentration)	Sets the peak rate for any diffusion-limited reaction
Absorption by individual receptors	$\frac{I}{I_0} = \frac{1}{\left(1 + \frac{\pi a}{Ns}\right)}$	I is diffusive current, a is sphere radius; N receptors of radius s distributed at random	Shows that a mere handful of receptors suffice to do very well!

see: **Random Walks in Biology**, expanded edition, Howard C. Berg, Princeton Univ. Press 1993.

Investigations on the Theory of Brownian Movement, Albert Einstein, with notes by R. Fürth, translated by A.D. Cowper, Dover Publications, 1956 (orig. published 1926). This theory constituted a significant part of Einstein's Ph.D. thesis!

Berg, H.C. & Purcell, E.M. (1977) "**Physics of Chemoreception**", *Biophysical Journal* **20**:193-219.